Linear Algebra Short Course

Lecture 1

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Some useful references

- Properties of vector spaces: Axler (1997, Ch. 1–2)
- Various spaces, more analysis than LA: Luenberger (1968, Ch. 2–3)
- Basic topology of metric spaces: Spivak (1965, Ch. 1), Mendelson (1990, Ch. 2), Rudin (1976, Ch. 1–2)

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1. **Basic framework of linear algebra**
2. **Properties and structure of linear spaces**
3. **Analysis on general vector spaces**
4. **Some important spaces**

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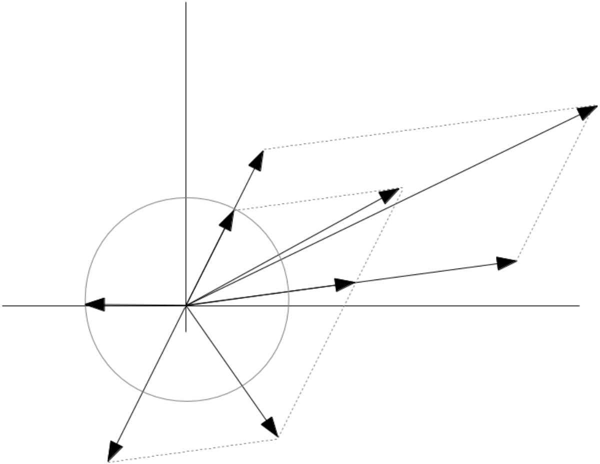
General motivations 1



In high school physics, we build intuition with objects called *vectors* and *scalars*, their properties of *length* and *angle*, and operations such as *rotations* and *translations*.

General motivations 2

Even at the elementary level, there is a natural progression:

Geometric (drawing arrows, etc.)

$\Downarrow$

Algebraic (using symbols, <u>defining operations</u>)

In basic "vector analysis," what sort of operations do we define?

V + V, S * V, projection, scalar product, vector product, . . .

Defining these for "vectors" in $\mathbb{R}^3$ and "scalars" in $\mathbb{R}$ is very fruitful!

But in mathematics, $\mathbb{R}^3$ is not the only space we're interested in.
What about $\mathbb{R}^n$ or even $\mathbb{R}^\infty$? Spaces of functions?

General motivations 3

What we do in linear algebra:

Define analogous operations on more general spaces

$\Downarrow$

Investigate their properties (i.e., prove interesting theorems)

We take an axiomatic approach to this.
Why is this a productive endeavour?

We only need to prove things once!

If $\mathbb{R}^3$ and $\mathbb{R}^\infty$ and $\mathcal{C}[a,b]$ all satisfy our axioms, and our proofs only use those axioms, proving it for one implies the others.

Enough heuristics, let's get started.

Basic framework: axioms for scalars 1

Our pool of scalars will be a “field” $\mathbb{F}$, defined below.

Defn. If non-empty set $\mathbb{F}$ with binary addition/multiplication operations defined such that (FA), (FM), and (FD) hold, we call $\mathbb{F}$ a **field**. Taking arbitrary $x, y, z \in \mathbb{F}$,

Addition axioms:

FA.1 $x, y \in \mathbb{F} \implies (x + y) \in \mathbb{F}$

FA.2 $x + y = y + x$

FA.3 $(x + y) + z = x + (y + z)$

FA.4 $\exists\, x' \in \mathbb{F}, x' + x = x, \forall\, x \in \mathbb{F}$. Denote 0.

FA.5 $\exists\, x' \in x', x' + x = 0$. Denote $-x$.

Basic framework: axioms for scalars 2

Multiplication and distribution axioms:

FM.1 $x, y \in \mathbb{F} \implies xy \in \mathbb{F}$

FM.2 $xy = yx$

FM.3 $(xy)z = x(yz)$

FM.4 $\exists\, x' \in \mathbb{F}, x'x = x, \forall\, x \in \mathbb{F}$. Denote 1.

FM.5 $\exists\, x' \in x', x'x = 1$. Denote $1/x$.

FD.1 $x(y+z) = xy + xz$

(*) Note $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ with usual operations are fields.

Basic framework: axioms for vectors 1

With underlying field $\mathbb{F}$ used as our source of scalars, now we discuss axioms for V, a set from which we get our vectors.

Defn. The non-empty set V, equipped with scalar multiplication and vector addition operations, is called a **(linear) vector space** on $\mathbb{F}$ when (VA), (VM), and (VD) below hold. Take $u, v, w \in V, \alpha, \beta, 0, 1 \in \mathbb{F}$.

Vector addition axioms:

VA.1 $u + v = v + u \in V$

VA.2 $(u + v) + w = u + (v + w)$

VA.3 $\exists\, \theta \in V, \theta + u = u, \forall\, u \in V$

VA.4 $\exists\, u' \in V, u + u' = \theta, \forall\, u \in V$. Denote $-u$.

Basic framework: axioms for vectors 2

Scalar multiplication and distribution axioms:

VM.1 $(\alpha\beta)u = \alpha(\beta u) \in V$

VM.2 $0u = \theta, \forall\, u \in V$

VM.3 $1u = u, \forall\, u \in V$

VD.1 $\alpha(u + v) = \alpha u + \alpha v$

VD.2 $(\alpha + \beta)u = \alpha u + \beta u$

Typically we just denote all additive identities by 0, so let $\theta = 0$.

That's all the groundwork we'll need to build our framework.

Basic properties

(*) Let $U_1, \ldots, U_n$ be vector spaces on common field $\mathbb{F}$. Then with the usual definition of the Cartesian product, verify $U_1 \times \cdots \times U_n$ is a vector space on $\mathbb{F}$.

(*) The following properties follow from our axioms on V:

VM.4 $\alpha 0 = 0$

VD.3 $(\alpha - \beta)x = \alpha x - \beta x$

VD.4 $\alpha(x - y) = \alpha x - \alpha y$

VC.1 $x + y = y + z \implies x = z$

VC.2 $\alpha \neq 0, \alpha x = \alpha y \implies x = y$

VC.3 $x \neq 0, \alpha x = \beta x \implies \alpha = \beta$

VM.5 $(-\alpha)x = \alpha(-x) = -(\alpha x)$

VM.6 $xy = 0 \implies x = 0$ or $y = 0$

(*) For vector space V, additive identity is always unique. Also, for each $v \in V$, additive inverse always unique.

Subspaces

Certain subsets of vector spaces will be of particular interest:

Defn. Let V be a vector space on field $\mathbb{F}$. Taking $X \subseteq V$, if

$$u + v \in X$$
$$\alpha u \in X$$

$\forall\, u, v \in X, \alpha \in \mathbb{F}$, then we call X a **(linear) subspace** of V.

Subspaces are thus the subsets closed under vector sums and scalar products.

(*) Note $X \subseteq V$ a subspace $\iff$ X is a vector space.

Basic framework of linear spaces

Example. (*) Given the "usual" algebraic operations, the following are linear spaces. Consider the operations and the vector space and field upon which they live.

- $\mathbb{F}^n$, given field $\mathbb{F}$.
- $\mathbb{F}^\infty = \{(x_1, x_2, \ldots) : x_i \in \mathbb{F}, i = 1, 2, \ldots\}$, given field $\mathbb{F}$.
- $\mathcal{P}(\mathbb{F}) = \{p : p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m\}$, given field $\mathbb{F}$, and coefficients $\alpha \in \mathbb{F}^m, m \geq 1$.
- $\{(x_1, x_2, \ldots) \in \mathbb{R}^\infty : x_n \to 0\}$
- $\{f : [a, b] \to \mathbb{R}; f$ continuous on $[a, b]\}$.

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Sums and decompositions into direct sums

Defn. Let V be a vector space, and $U_1, \ldots, U_n \subset X$ be subsets. We define the **sum** of these sets respectively as

$$U_1 + \cdots + U_n := \{u_1 + \cdots + u_n : u_i \in U_i, 1 \leq i \leq n\}.$$

If for every $z \in V$, we have that each representation

$$z = u_1 + \cdots + u_n, \text{ where } u_i \in U_i, 1 \leq i \leq n$$

is unique, then we write $V = U_1 \oplus \cdots \oplus U_n$, the **direct sum** of the U_i.

What operations preserve linearity?

Given a sum, under what conditions is it a direct sum decomposition?

Basic properties of sums

Let $U, W, U_1, \ldots, U_n \subset V$ be subspaces of vector space V.
(*) Then,

$$V = U_1 \oplus \cdots \oplus U_n \iff U_1 + \cdots + U_n = V$$
$$\text{and } 0 = u_1 + \cdots + u_n \text{ uniquely } u_i = 0.$$

(*) This leads to a nice corollary,

$$V = U \oplus W \iff V = U + W \text{ and } U \cap W = \{0\}.$$

(**) Sums and intersections (happily) preserve linearity:

$$U + W \text{ and } U \cap W \text{ are subspaces.}$$

(*) Note this extends to arbitrary sums/intersections of subspaces.
(*) Unions need not preserve linearity.
(*) For subspaces $U, W \subset V$, have that $[U \cup W] = U + W$.

Linear combinations

Defn. Given vector space V on $\mathbb{F}$, for any $m \geq 1$ elements $x_1, \ldots, x_m \in V$ and $\alpha \in \mathbb{F}^m$, we call

$$\alpha_1 x_1 + \cdots + \alpha_m x_m$$

a **linear combination** of these elements.

(*) Note we only defined pairwise sums, but the axioms imply this is notation is unambiguous.

(*) If $S \subset V$ is a subspace then S closed under linear combinations.

Defn. For subset $T \subset V$, define

$$[T] := \{\text{all linear combinations of elements in } T\}$$

called the **subspace generated by** T, or the “span” of T.

(*) Validate this defn; $[T]$ a subspace of V, the “smallest” containing T.

Translations of linear spaces

Example. Consider the hyperplane $H \subset \mathbb{R}^n$ given by

$$H = \{x : \alpha^T x = b\}, b \neq 0.$$

(*) While defined by a linear relation, this is *not* a subspace.

Defn. Any $W \subset V$ containing all lines through any two points we call an **affine** set. That is, for $u, v \in W$, have

$$\lambda u + (1 - \lambda)v \in W, \ \forall \lambda \in \mathbb{R}.$$

The **affine hull** of a set $T \subset V$ is defined

$$\operatorname{aff} T := \cap W_i$$

intersection over all affine sets $W_i \subset V$ s.t. $T \subset W_i$.

(*) Validate this definition; $\operatorname{aff} T$ is well-defined, is affine.
(*) Every affine set is a translation of a subspace.

Linear dependence, dimension, basis

Foundational concepts for analysis of linear spaces.
Assume V a vector space on $\mathbb{F}$.

Defn. Take non-empty $S \subset V$. We say $x \in V$ is **linearly dependent** on S if x is a linear combination of elements of S. Equivalently,

$$x \text{ is linearly dependent on } S \iff x \in [S].$$

If this doesn't hold, say x is **linearly independent** of S. Analogously, say $S \subset V$ is a **linearly independent set** of vectors when

$$u \text{ lin indep of } S \setminus \{u\}, \ \forall\, u \in S.$$

(*) For any finite set $\{x_1, \ldots, x_n\} \subset V$, the following is key:

$$\{x_1, \ldots, x_n\} \text{ is linearly indep.} \iff \sum_{i=1}^{n} \alpha_i x_i = 0 \text{ implies all } \alpha_i = 0.$$

Linear dependence, dimension, basis

Defn. We call a vector space V **finite dimensional** if there exists a *finite* subset $B \subset V$ such that $[B] = V$. If no such finite subset exists, call V **infinite-dimensional**.

If B is linearly independent, call B a **basis** of V.

(*) If $\{v_1, \ldots, v_n\}$ a basis of V, then every $v \in V$ may be uniquely represented in the form

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

(*) In fact, the uniqueness of this representation *characterizes* $\{v_1, \ldots, v_n\}$ as a basis.

Does a basis B always exist?
Can we make any statements about its length $|B|$?
Is dimensionality monotonic in some sense?

Linear dependence, dimension, basis

(**) The following foundational results are valid:

Let V be a finite-dim vector space. Then,

- $\exists\, B \subset V$ s.t. B a basis of V
- if B, C bases of V, then $|B| = |C|$.
- $S \subset V$ a subspace $\implies$ S finite-dim.
- $S \subset V$ a subspace $\implies$ $|B_S| \leq |B_V|$, each respective bases.

These results completely motivate a (now well-definable) dimension notion.

Linear dependence, dimension, basis

Defn. Let V be a finite-dimensional vector space. Define the **dimension** of V by $\dim V := |B|$, where B is any basis of V. If V is infinite-dim, let $\dim V := \infty$, if $V = \{0\}$, let $\dim V = 0$.

(*) Clearly we have our monotonicity, where subspace $S \subset V$ satisfies

$$\dim S \leq \dim V.$$

As well, one may show that if we know $\dim V = n$, we only need one more piece of information to validate a given $\{v_1, \ldots, v_n\} \subset V$ as a basis, since

$$[\{v_1, \ldots, v_n\}] = V \implies \{v_1, \ldots, v_n\} \text{ a basis of } V$$
$$\{v_1, \ldots, v_n\} \text{ lin indep} \implies \{v_1, \ldots, v_n\} \text{ a basis of } V.$$

Linear dependence, dimension, basis

Let $S, T, U, U_1, \dots, U_n \subset V$ be subspaces, $\dim V < \infty$.

(**) Handily, it is possible to verify

$$\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$$
$$\dim(U_1 + \cdots + U_n) \leq \dim U_1 + \cdots + \dim U_n.$$

(*) Unfortunately, the following *does not* hold in general:

$$\begin{aligned}\dim(S+T+U) = {} & \dim S + \dim T + \dim U - \dim(S \cap T) \\ & - \dim(S \cap U) - \dim(T \cap U) + \dim(S \cap T \cap U).\end{aligned}$$

Linear dependence, dimension, basis

Interestingly, the "structure" of vector space V, $\dim V < \infty$ is captured well by its dimension.

(*) For any vector space V, $\dim V = n$, there exist one-dim $V_1, \ldots, V_n$ s.t.

$$V = V_1 \oplus \cdots \oplus V_n.$$

(*) Also, if $S \subset V$ is a subspace, then

$$\dim S = \dim V \implies S = V.$$

(**) Taking subspaces $U_1, \ldots, U_n \subset V$,

$$V = U_1 + \cdots + U_n \text{ and } \dim V = \sum_{i=1}^{n} \dim U_i \iff V = U_1 \oplus \cdots \oplus U_n.$$

This proof is another straightforward exercise.

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Brief analysis review

Linear spaces are *ubiquitous* in mathematics. To introduce some important examples, let's review a few basic concepts from analysis.

Defn. Given set X, a function $d : X \times X \to \mathbb{R}_+$ is called a **metric** if $\forall\, x, y, z \in X$,

M.1 $d(x, y) \geq 0,$ with equality iff $x = y$

M.2 $d(x, y) \leq d(x, y) + d(y, z)$

M.3 $d(x, y) = d(y, x)$

We call X equipped with a metric d a **metric space**.

(**) The following are metric spaces:

- $\mathbb{R}^n$ with $d(x, y) := (\sum_{i=1}^n (x_i - y_i)^2)^{1/2}$
- $\mathbb{R}^n$ with $d(x, y) := \max_i |x_i - y_i|$
- $\mathcal{C}[a, b]$ with $d(f, g) := \int_a^b |f(t) - g(t)|\, dt$
- $d(f, g) := \sup_{a \leq x \leq b} |f(x) - g(x)|, f, g$ bounded on $[a, b] \subset \mathbb{R}$.

Brief analysis review

Defn. Denoting the ε-radius ball at x_0 in metric space X by

$$\varepsilon B(x_0) := \{x \in V : d(x, x_0) < \epsilon\},$$

for any $S \subset X$, call $u \in S$ an **interior point** if $\exists\, \varepsilon > 0$ s.t.

$$\varepsilon B(u) \subset S.$$

Denote all such points by $\operatorname{int} S$, the **interior** of S. If $S = \operatorname{int} S$, call S an **open** subset of X.

Call $p_0 \in V$ a **limit point** of $S \subset V$ if $\forall\, \delta > 0$,

$$\exists\, x \in S, x \neq p_0, \text{ s.t. } x \in \delta B(p_0).$$

If S^* is all limit points of S, call $\bar{S} := S \cup S^*$ the **closure** of S. Call S a **closed** subset of X when $S = \bar{S}$.

Vector "magnitude" in linear spaces

Defn. If V a vector space on field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we call a map $x \mapsto \|x\| \in \mathbb{R}_+$ a **norm** if $\forall\, u, v \in V, \alpha \in \mathbb{F}$,

N.1 $\|u\| > 0$ for $u \neq 0$, and $\|0\| = 0$.

N.2 $\|u + v\| \leq \|u\| + \|v\|$

N.3 $\|\alpha u\| = |\alpha| \|u\|$

We call V equipped with a norm a **normed linear space**.

(*) Note any norm on V induces a valid metric on V.

(*) What about the converse? Consider a "reverse indicator" metric.

(*) $\mathcal{C}[a, b]$ with $\|f\| := \sup_{a \leq x \leq b} |f(x)|$ is normed vec space.

Convergence in normed linear spaces

Let $(X, \|\cdot\|)$ be a normed vector space, and (x_n) a sequence of vectors $x_1, x_2, \ldots \in X$.

Defn. We say a sequence (x_n) **converges** to $x \in X$ (in the norm $\|\cdot\|$), denoted $x_n \to x$,

$$\text{whenever } \lim_{n\to\infty} \|x_n - x\| = 0,$$

noting $(\|x_n - x\|)$ is a sequence of real numbers.

(*) The limits of convergent sequences are unique.
(*) $S \subset X$ is closed $\iff$ Every sequence (x_n) in S converges in S.

Continuous maps in normed linear spaces

Once again say $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$, are normed vector spaces.

Defn. Continuity of $f : X \to Y$ extends in the natural way, of course. Namely, f is **continuous** at $x_0 \in X$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\|x - x_0\|_X < \delta \implies \|f(x) - f(x_0)\|_Y < \varepsilon.$$

Clearly this depends on both norms.

(*) f is continuous at $x_0 \iff x_n \to x_0$ implies $f(x_n) \to f(x_0)$.

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Banach spaces

Defn. We call (x_n) a **Cauchy sequence** if $\forall\, \varepsilon > 0$, $\exists\, N_0 < \infty$ s.t.

$$m, n \geq N_0 \implies \|x_n - x_m\| < \varepsilon.$$

If all Cauchy sequences on X converge, we say X is **complete** (in norm $\|\cdot\|$). We call a complete normed linear space a **Banach space**.

The "Cauchy condition" is precisely why Banach spaces are nice.

(*) All Cauchy sequences are bounded in norm $\|\cdot\|$.
(*) All convergent sequences are Cauchy.
(*) $\mathcal{C}[a, b]$ with $\|f\| := \sup_{a \leq x \leq b} |f(x)|$ is Banach.
(**) $\mathcal{C}[a, b]$ with $\|f\| := \int_a^b |f(x)|\, dx$ is *not* Banach.

More on Banach spaces

(*) If X, Y are Banach, the “usual” product space $(X \times Y, \|\cdot\|)$ with $\|\cdot\| := \|\cdot\|_X + \|\cdot\|_Y$ is Banach.

(*) Let X be Banach; subset $S \subset X$ is complete $\iff$ S is closed.

(**) Another key result: if X is a normed linear space, for $S \subseteq X$,

$$\dim S < \infty \implies S \text{ is complete.}$$

Example: ℓ_p space, $1 \le p \le \infty$

Here we introduce “the” classical Banach space.

Defn. Define **ℓ_p-space** for $1 \le p < \infty$ by

$$\ell_p := \left\{ (x_1, x_2, \ldots) : \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}.$$

The norm of interest is of course $\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$.

For the $p = \infty$ case, we consider *bounded* sequences, and intuitively we define $\|x\|_\infty := \sup |x_i|$.

Example: $L_p(\Omega, \mathcal{A}, \mathbf{P})$ space, $1 \leq p \leq \infty$

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space.
Defn. We define L_p**-space** on $(\Omega, \mathcal{A}, \mathbf{P})$ for $1 \leq p < \infty$ by

$$L_p := \left\{ h : \mathbf{E}\,|h|^p = \int_\Omega |h(\omega)|^p \ \mathbf{P}(d\omega) < \infty \right\},$$

and the usual norm is $\|h\|_p := (\mathbf{E}\,|h|^p)^{1/p}$.

For $p = \infty$ case, consider bounded functions $\sup_\omega |h(\omega)| < \infty$.

Minor complication:
Even if $g, h \in L_p$ are $g \neq h$, we might have $g = h$ a.e. $[\mathbf{P}]$.

(*) For those familiar with Lebesgue integration, why is defining $\|h\|_\infty = \sup_\omega |h(\omega)|$ inadvisable? Any ideas for an alternative?

Both ℓ_p and L_p are Banach

Very important, classic results, proofs are critical for any serious student of analysis (out of scope here).

The basic flow (ℓ case) is:

(1) For $x \in \ell_p, y \in \ell_q$ where $1/p + 1/q = 1$, prove Hölder's inequality,

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q.$$

(2) Using Hölder, for $x, y \in \ell_p$ prove Minkowski's inequality:

$$x + y \in \ell_p \text{ and } \|x + y\|_p \leq \|x\|_p + \|y\|_p,$$

namely the triangle inequality. Definiteness easy on ℓ_p, but requires thought on L_p.

(3) Then just need completeness. ℓ_p is basic analysis, L_p requires some Lebesgue theory.

Inner product and Hilbert space

Consider vector space V on field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Defn. Call $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ an **inner product** on V if $\forall\, u, v, w \in V, \alpha \in \mathbb{F}$,

IP.1 $\langle u, v \rangle = \overline{\langle v, u \rangle}$

IP.2 $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$

IP.3 $\langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$

IP.4 $\langle u, u \rangle \geq 0$, and $\langle u, u \rangle = 0 \iff u = 0$.

(*) Additivity holds in both arguments. Also, $\langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle$.

(**) We'll later see that an IP on V induces a norm on V (Lec 4).

Defn. Call $(V, \langle \cdot, \cdot \rangle)$ an **inner product space**. A complete IP space is called **Hilbert space**. Much more later.

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